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MODIFIED FIXED POINT AND COMMON FIXED POINT THEOREMS IN CONE b-METRIC SPACES WITH PARTIALLY ORDERED SET

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Abstract

The aim of this paper is to prove that fixed point theorem and common fixed point theorem on ordered Cone b-metric spaces [18] the main results of this paper provide extensions as well as substantial generalizations and improvements of several well known results in the recent literature. Also, we introduce some examples to support the usability of our results are applicable on ordered cone b-metric spaces.

1. Introduction

Fixed point theory has captivated many researchers since 1922 with the commendable Banach fixed point theorem. This theorem supplies a method for solving diverseness of applied problems in mathematical sciences and with computer sciences, engineering. A

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large published works on this subject exists, and this is a dynamic area of research at present time. Banach contraction principle has been generalized in diverse directions in different spaces by mathematicians over the years; we refer to [1, 5, 11, 21, 22] and (references therein) for more detail on this and related topic.

In modern time, fixed point theory has evolved quickly in partially ordered cone metric spaces; that is, cone metric spaces equipped with a partial ordering, for some new results in ordered metric spaces [13]. Preliminary Conclusion in this bearing was given by Altun and Durmaz [9] under the condition of normality for cones. Then, Altun et al. [10] concluded the results of Altun and Durmaz [9] by excluding the assumption of normality condition for cones. Afterward, several authors have studied fixed point and common fixed point problems in ordered cone metric spaces; for more details [3, 8, 17, 19, 20, 23, 24, 25].

In 2011, Hussain and Shah [16] presented cone b-metric spaces and con metric spaces for some new finding in b-metric spaces [6]. They not only devised some topological properties in such spaces but also improvised some current results about KKM mappings in the setting of a cone b-metric space.

After some time, many researchers have been inspired to manifest fixed point theorems as well as common fixed point theorems for two or more mappings on cone bmetric spaces by the initial work of Hussain and Shah [16] and [2, 7, 14, 15].

2. Preliminaries and Definitions

Theorem 2.1 [9] : Let $(X, \subseteq)$ be a partially ordered set, suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete, and let P be a normal cone with normal constant K. Let $f : X \to X$ be a continuous and increasing mapping with respect to $\subseteq$. Suppose that the following three conditions hold:

(i) there exists $k \in [0, 1)$ such that $d(fx, fy) \preceq d(x, y)$ for all $x, y \in X$ with $y \subseteq x$;

(ii) there exists $x_0 \in X$ such that $x_0 \subseteq fx_0$.

Then f has a fixed point in X.

Theorem 2.2 [10] : Let $(X, \subseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete over a solid cone P.

Let $f : X \to X$ be a continuous and increasing mapping with respect to $\subseteq$. Suppose that the following two conditions holds:

(i) there exist $k, l, r \in [0.1]$ with $k + 2l + 2r < 1$ such that

$$d(fx, fy) \preceq kd(x, y) + l(d(fx, x) + d(fy, y)) + r(d(fx, y) + d(fy, x)) \quad (1)$$

for all $x, y \in X$ with $y \subseteq x$;

(ii) there exists $x_0 \in X$ such that $x_0 \subseteq fx_0$.

Then f has a fixed point in X.

Theorem 2.3 [10] : Let $(X, \subseteq)$ be a partially ordered set and suppose that there exists a cone metric b in X such that the cone metric space (X, d) is complete over a solid cone P.

Let $f : X \to X$ be a increasing mapping with respect to $\subseteq$ Suppose that the following three conditions holds:

(i) there exist $k, l, r \in [0, 1)$ with $k2l + 2r < 1$ such that

$$d(fx, fy) \leq kd(x, y) + l(d(fx, x) + d(fy, y) + r(d(fx, y) + d(fy, x)) \quad (2)$$

for all $x, y \in X$ with $y \subseteq x$;

(ii) there exists $x_0 \in X$ such that $x_0 \subseteq fx_0$;

(iii) if an increasing sequence $\{x_n\}$ converges to x in X, then $x_n \subseteq x$ for all n.

Then f has a fixed point in X.

In the same paper, they also presented the following two common fixed point results in ordered cone metric spaces.

Theorem 2.4 [10] : Let $(X, \subseteq)$ be a partially ordered set and suppose that there exists a cone metric b in X such that the cone metric space (X, d) is complete over a solid cone P.

Let $f, g : X \to X$ be two weakly increasing mappings with respect to $\subseteq$. Suppose that the following three conditions holds:

(i) there exist $k, l, r \in [0, 1)$ with $k + 2l + 2r < 1$ such that

$$(fx, gy) \preceq kd(x, y) + l(d(x, fx) + d(y, gy)) + r(d(y, fx) + d((x, gy)) \tag{3}$$

for all comparative $x, y \in X$;

(ii) f or g is continuous.

Then f and g have a common fixed point $x^* \in X$.

Theorem 2.5 [10] : Let $(X, \subseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete over a solid cone P.

Let $f, g : X \to X$ be two weakly increasing mappings with respect to $\subseteq$. Suppose that the following three conditions holds:

(i) there exist $k, l, r \in [0, 1)$ with $k + 2l + 2r < 1$ such that

$$d(fx, gy) \preceq kd(x, y) + l(d(x, fx) + d(y, gy)) + r(d(y, fx) + d(x, gy)) \tag{4}$$

for all comparative $x, y \in X$;

(ii) if an increasing sequence $\{x_n\}$ converges to x in X, then $x_n \subseteq x$ for all n.

Then f and g have a common fixed point $x^* \in X$.

In this paper, we prove some fixed point and common fixed point theorems on ordered cone b-metric spaces. Our results extend and generalize several well-known comparable results in the literature to ordered cone b-metric spaces. Throughout this paper, we do not impose the normality condition for the cones, but the only assumption is that the cone P is solid, that is, $P \neq 0$.

The following definitions and results shall be needed in the sequel.

Let E be a real Banach space and θ denotes the zero element in E. A cone P is a subset of E such that

(1) P is nonempty closed set and $P \neq \{\theta\}$;

(2) if a, b are nonnegative real numbers and $x, y \in P$, then $ax + by \in P$;

(3) $x \in P$ and $-x \in P$ imply $x = \theta$.

For any cone $P \subset E$, the partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation of $\prec$ stands for $x \preceq y$ but $x \neq y$. Also, we use $x \ll y$ to indicate that $y - x \in \text{int } P$, where int P denotes the interior of P. A cone P is called normal if there exists the number K such that

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leq \|y\|. \tag{5}$$

for all $x, y \in E$. The least positive number K satisfying the above condition is called the normal constant of P.

Definition 2.6 [16] : Let X be a nonempty set and E a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone P. A vector-valued function $d : X \times X \to E$ is said to be a cone b-metric function on X with the constant $s \geq 1$ if the following conditions are satisfied.

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \preceq s(d(x, y) + d(y, z))$ for all $x, y, z \in X$.

Then pair (X, d) is called a cone b-metric space (or a cone metric type space); we shall use the first mentioned term.

Observe that if $s = 1$, then the ordinary triangle inequality in a cone metric space is satisfied; however, it does not hold true when $s > 1$. Thus the class of cone bb-metric spaces is effectively larger than that of the ordinary cone metric spaces. That is, every cone metric space is a cone b-metric space, but the converse need not be true. The following examples show the above remarks.

Example 2.7 : Let $X = \{-1, 0, 1\}$, $E = R^2$ and $P = \{(x, y) : x \geq 0, y \geq 0\}$.

Define $d : X \times X \to P$ by $d(x, y) = d(y, x)$ for all E$x, y \in X$, $d(x, x) = \theta, x \in X$ and $d(-1, 0) = (3, 3), d(-1, 1) = d(0, 1) = (1, 1)$. Then (X, D) is a complete cone b-metric space but the triangle inequality is not satisfied. Indeed, we have that $(-1, 1) + (1, 0) = (1, 1) + (1, 1) = (2, 2) \prec (3, 3) = (-1, 0)$. It is not hard to verify that $s = 3/2$.

Example 2.8 : Let $X = R, E = R^2$ and $P = \{(x, y) \in: x \geq 0, y \geq 0\}$. Define $X \times X \to E$ by $(x, y) = (|x - y|^2, |x - y|^2)$. Then, it is easy to see that (X, d) is a cone b-metric space with the coefficient $s = 2$. But it is not a cone metric spaces since the triangle inequality is not satisfied.

Definition 2.9 [16] : Let (X, d) be a cone b-metric space, $\{x_n\}$ a sequence in X and $x \in X$.

(1) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $(x_n, x) \ll c$ for all $n > N$ then x_n is said to be convergent and x is the limit of $\{x_n\}$. One denotes this by $x_n \to x$.

(2) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $(x_n, x_n) \ll c$ for all $n, m > N$, then $\{x_n\}$ is called complete if every Cauchy sequence in X.

(3) A cone metric space (X, d) is called complete if every Cauchy sequence in X is convergent.

The following lemma is useful in our work.

Lemma 2.10 [4] :

(1) If E is a real Banach space with a cone P and $a \preceq \lambda a$ where $a \in P$ and $0 < \lambda < 1$, then $a = \theta$.

(2) If $c \in \text{int } P, \theta \preceq a_n$ and $a_n \to \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.

(3) If $a \preceq c$ and $b \ll c$, then $a \ll c$.

(4) If $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.

3. Main Results

3.1. Fixed Point Results

In this section, we prove some fixed point theorems on ordered cone b-metric space. We begin with a simple but a useful lemma.

Lemma 3.2 : Let $\{x_n\}$ be a sequence in a cone b-metric space (X, d) with the coefficient $s \geq 1$ relative to a solid cone P such that

$$d(x_n, x_{n-1}) \preceq hd(x_{n-1}, x_n). \tag{6}$$

where $h \in [0, 1/s)$ and $n = 1, 2, \cdots$. Then $\{x_n\}$ is a Cauchy sequence in (X, d).

Proof : Let $m > n \geq 1$. It follows that

$$d(x_n, x_m) \preceq s\ d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{m-n} d(x_{m-1}, x_m). \tag{7}$$

Now, (6)and $sh < 1$ imply that

$$\begin{aligned} d(x_n, x_m &\preceq s\ d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{m^-n} d(x_{m-1}, x_m) \\ &\preceq s\ h^n d(x_0, x_1) + s^{m-n} d(x_0, x_1) + \cdots + s^{m^-n} h^{m^{-1}} d(x_0, x_1) \\ &= (sh^n + s^2 h^{n+1}) + \cdots + s^{m^-n} h^{m^{-1}} d(x_0, x_1) \\ &= sh^n (1 + sh + (sh)^2 + \cdots + (sh)^{m^-n^{-1}} d(x_0, x_1) \\ &\preceq \frac{sh^n}{1 - sh} d(x_0, x_1) \to \theta \quad \text{as} \quad n \to \infty. \end{aligned} \tag{8}$$

According to Lemma 2.10(2), and for any $c \in E$ with $c \gg \theta$, there exists $N_0 \in N$ such that for any $n > N_0, (sh^n/(1 - sh))d(x_0, x_1) \ll c$. Furthermore, from (8) and for any $m > n > N_0$, Lemma 2.10(3) shows that

$$d(x_n, x_m) \ll c. \tag{9}$$

Hence, by Definition 2.9(2) $\{x_n\}$ is a Cauchy sequence in X.

Theorem 3.3 : Let $(X, \subseteq)$ be a partially ordered set and suppose that there exists a cone b-metric d in X such that the cone b-metric space (X, d) is complete with the coefficient $s \geq 1$ relative to a solid cone P.. Let $f : X \to X$ be a continuous and increasing mapping with respect to $\subseteq$. Suppose that the following three conditions holds:

(i) there exist $i = 1, \ldots 6$, such that $2s\alpha_1 + 2\alpha_6 + (s+1)(\alpha_2 + \alpha_3) + (s^2 + s)(\alpha_4 + \alpha_5) < 2$ with $\sum_{i=1}^{6} \alpha_i < 1$,

$$\begin{aligned} d(fx, fy) &\preceq \alpha_1 d(x, y) + \alpha_2 d(fx, x) + \alpha_3 d(fy, y) \\ &\quad \alpha_4 d(fx, y) + \alpha_5 d(fy, x) + \alpha_6 d(fx, fy) \end{aligned} \tag{10}$$

for all $x, y \in X$ with $y \subseteq x$;

(ii) there exists $x_0 \in X$ such that $x_0 \subseteq fx_0$.

Then f has a fixed point $x^* \in X$.

Proof : If $x_0 = fx_0$, then the proof is finished. Suppose that $x_0 \neq fx_0$. Since $x_0 \subseteq fx_0$ and f is increasing with respect to $\subseteq$, we obtain by induction that $x_0 \subseteq fx_0 = x_1 \subseteq f^1x_0 = x_2 \subseteq \cdots \subseteq f^{n-1}x_0 = x_n \subseteq f^n x_0 = x_{n+1}$. Then we have

$$\begin{aligned}
d(x_{n+1}, x_n) &= d(f^n x_0, f^{n-1}x_0) \\
&= d(f(f^{n-1}x_0), f(f^{n-2}x_0)) \\
&\preceq \alpha_1 d(f^{n-1}x_0, f^{n-2}x_0) + \alpha_2 d(f^n x_0, f^{n-1}x_0) + \alpha_3 d(f^{n-1}x_0, f^{n-2}x_0) \\
&\quad +\alpha_4 d(f^n x_0, f^{n-2}x_0) + \alpha_5 d(f^{n-1}x_0, f^{n-1}x_0) + \alpha_6 d(f^n x_0, f^{n-1}x_0) \\
&= \alpha_1 d(x_n, x_{n-1}) + \alpha_2 d(x_{n+1}, x_n) + \alpha_3 d(x_n, x_{n-1}) + \alpha_4 d((x_{n+1}, x_{n-1}) \\
&\quad +\alpha_5 d(x_n, x_n) + \alpha_6 d(x_{n+1}, x_n) \\
&\preceq \alpha_1 d(x_n, x_{n-1}) + \alpha_2 d(x_{n-1}, x_n) + \alpha_3 d(x_n, x_{n-1}) + s\alpha_4(d(x_{n-1}, x_n) \\
&\quad +d(x_n, x_{n+1})) + \alpha_6 d(x_{n+1}, x_n). \qquad (11)
\end{aligned}$$

Then, one can assert that

$$d(x_{n+1}, x_n) \preceq (\alpha_1 + \alpha_3 + s\alpha_4)d(x_n, x_{n-1}) + (\alpha_2 + s\alpha_4 + \alpha_6)d(x_{n+1}, x_n). \qquad (12)$$

On the other side, we have

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(f^{n-1}x_0, f^n x_0) \\
&= d(f(f^{n-2}x_0), f(f^{n-1}x_0)) \\
&\prec \alpha_1 d(f^{n-2}x_0, f^{n-1}x_0) + \alpha_2 d(f^{n-1}x_0, f^{n-2}x_0) \\
&\quad +\alpha_3 d(f^n x_0, f^{n-1}x_0) + \alpha_4 d(f^{n-1}x_0, f^{n-1}x_0) \\
&\quad +\alpha_5 d(f^n x_0, f^{n-2}x_0) + \alpha_6 d(f^{n-1}x_0, f^n x_0) \\
&= \alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_n, x_{n-1}) + \alpha_3 d(x_{n+1}, x_n) \\
&\quad +\alpha_4 d(x_n, x_n) + \alpha_5 d(x_{n+1}, x_{n-1}) + \alpha_6 d(x_n, x_{n+1}) \\
&\preceq \alpha_1 d(x_n, x_{n-1}) + \alpha_2 d(x_n, x_{n-1}) + \alpha_3 d(x_{n+1}, x_n) \\
&\quad +s\alpha_5(d(x_{n+1}, x_n) + d(x_n, x_{n-1}) + \alpha_6 d(x_n, x_{n+1}). \qquad (13)
\end{aligned}$$

Then, one can assert that

$$d(x_{n+1}, x_n) \preceq (\alpha_1 + \alpha_2 + s\alpha_5)d(x_n, x_{n-1}) + (\alpha_3 + \alpha_6 + s\alpha_5))d(x_{n+1}, x_n). \qquad (14)$$

Adding (12) and (14), we get

$$d(x_{n+1}, x_n) \preceq \frac{2\alpha_1 + \alpha_2 + \alpha_3 + s\alpha_4 + s\alpha_5}{2 - (\alpha_2 + \alpha_3 + 2\alpha_6 + s\alpha_4 + s\alpha_5)} d(x_n, x_{n-1}) = \lambda(x_n, x_{n-1}) \qquad (15)$$

where $\lambda = (2\alpha_1 + \alpha_2 + \alpha_3 + s\alpha_4 + s\alpha_5)(2 - (\alpha_2 + \alpha_3 + 2\alpha_6 + s\alpha_4 + s\alpha_5)) < 1/s$. According to Lemma 3.2, we have $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$. Since f is continuous, then $x^* = \lim x_{n+1} = \lim f^n x_0 = \lim f(f^{n-1}x_0) = f(\lim f^{n-1}x_0) = f(\lim x_n) = (x^*)$. Therefore, x^* is a fixed point of f.

Theorem 3.4 : Let $(X, \subseteq)$ be a partially ordered set and suppose that there exists a cone b-metric d in X such that the cone b-metric space (X, d) is complete with the coefficient $s \geq 1$ relative to a solid cone P. Let $f : X \to X$ be a increasing mapping with respect to $\subseteq$. Suppose that the following three conditions holds:

(i) there exist $\alpha_i, i = 1, \cdots, 6$, such that $2s\alpha_1 + 2\alpha_6 + (s+1)(\alpha_2 + \alpha_3) + (s^2 + s)(\alpha_4 + \alpha_5) < 2$ with $\sum_{i=1}^{6} \alpha_i < 1$.

$$\begin{aligned} d(fx, fy) &\preceq \alpha_1 d(x, y) + \alpha_2 d(fx, x) + \alpha_3 d(fy, y) \\ &\quad + \alpha_4 d(fx, y) + \alpha_5 d(fy, x) + \alpha_6 d(fx, fy) \end{aligned} \qquad (16)$$

for all $x, y \in X$ with $y \subseteq x$;

(ii) there exists $x_0 \in X$ such that $x_0 \subseteq fx_0$;

(iii) if an increasing sequence $\{x_n\}$ converges to x in X, then $x_n \subseteq x$ for all m.

Then f has a fixed point $x^* \in X$.

Proof : As in the Theorem 3.3, we can construct an increasing sequence $\{x_n\}$ and prove that there exists $x^* \in X$ such that $x_n \to x^*$. Now, condition (iii) implies $x_n \subseteq x^*$ for all n.

Therefore, we can use condition (i) and so

$$\begin{aligned} d(fx_n, fx^*) &\preceq \alpha_1 d(x_n, x^*) + \alpha_2 d(fx_n, x_n) + \alpha_3 d(fx^*, x^*) \\ &\quad + \alpha_4 d(fx_n, x^*) + \alpha_5 d(fx^*, x_n) + \alpha_6 d(fx_n, fx^*). \end{aligned} \qquad (17)$$

Taking $n \to \infty$, we have $(x^*, fx^*) \preceq (\alpha_3 + \alpha_5 + \alpha_6)(x^*, fx^*) d(x^*, fx^*)$. Since $(\alpha_3 + \alpha_5 + \alpha_6) < 1$, Lemma 2.10 (1) shows that $(x^*, fx^*) = \theta)$, that is $x^* = fx^*$. Therefore x^* is a fixed point of f.

3.1 Common Fixed Point Results

Now, we give two common fixed point theorems on ordered cone b-metric spaces. We need the following definitions.

said to be weakly increasing if $fx \subseteq gfx$ and $gx \subseteq fgx$ hold for all $x \in X$.

Theorem 3.5 : Let $(X, \subseteq)$ be a partially ordered set and suppose that there exists a cone b-metric d in X such that the cone b-metric space (X, d) is complete with the coefficient $s = 1$ relative to a solid cone P. Let $f, g : X \to X$ be two weakly increasing mappings with respect to $\subseteq$. Suppose that the following three conditions holds:

(i) there exist $\alpha_i, i = 1, \cdots, 6$, such that $2s\alpha_6 + (s+1)(\alpha_2 + \alpha_3) + (s^2 + s)(\alpha_4 + \alpha_5) < 2$ with $\sum_{i=1}^{6} \alpha_i < 1$,

$$\begin{aligned} d(fx, gy) \ \preceq \ & \alpha_1 d(x, y) + \alpha_2 d(x, fx) + \alpha_3 d(y, gy) \\ & + \alpha_4 d(y, fx) + \alpha_5 d(x, gy) + \alpha_6 d(fx, gy) \end{aligned} \tag{18}$$

for all comparative $x, y \in X$;

(ii) f or g is continuous.

Then f and g have a common fixed point $x^* \in X$.

Proof : Let x_0 be an arbitrary point of X and define a sequence $\{x_n\}$ in X as follows: $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n > 0$. Note that, since f and g are weakly increasing, we have

$$x_1 = fx_0 \subseteq gfx_0 = gx_1 = x_2 \ \text{ and } \ x_2 = gx_1 \subseteq fgx_1 = fx_2 = x_3$$

and continuing this process we have $x_1 \subseteq x_2 \subseteq \cdots \subseteq x_n \subseteq x_{n+1} \subseteq \cdots$. That is, the sequence $\{x_n\}$ is non decreasing. Now, since x_{2n} and x_{2n+1} are comparative, we can use the inequality (18), and then we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) \ = \ & (fx_{2n}, gx_{2n+1}) \\ \preceq \ & \alpha_1 d(x_{2n}, x_{2n+1}) + \alpha_2 d(x_2, fx_{2n}) \\ & + \alpha_3 d(x_{2n+1}, gx_{2n+1}) + \alpha_4 d(x_{2n+1}, fx_{2n}) \\ & + \alpha_5 d(x_{2n}, gx_{2n+1}) + \alpha_6 d(fx_{2n}, gx_{2n+1}) \end{aligned}$$

$$\begin{aligned}
&\preceq \alpha_1 d(x_{2n}, x_{2n+1}) + \alpha_2 d(x_{2n}, x_{2n+1}) \\
&\quad +\alpha_3 d(x_{2n+1}, x_{2n+2}) + \alpha_4 d(x_{2n+1}, x_{2n+1}) \\
&\quad +\alpha_5 d(x_{2n}, x_{2n+2}) + \alpha_6 d(x_{2n+1}, x_{2n+2}) \\
&\preceq (\alpha_1+\alpha_2) d(x_{2n}, x_{2n+1}) + \alpha_3 (d)(x_{2n+1}, x_{2n+2}) \\
&\quad +s\alpha_5 (d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + \alpha_6 d(x_{2n+1}, x_{2n+2}) \\
&= (\alpha_1+\alpha_2+s\alpha_5) d(x_{2n}, x_{2n+1}) \\
&\quad +(\alpha_3+\alpha_6+s\alpha_5)(x_{2n+1}, x_{2n+2}). \qquad (19)
\end{aligned}$$

Hence,

$$(1-(\alpha_3+\alpha_6+s\alpha_5)) d(x_{2n+1}, x_{2n+2}) \preceq (\alpha_1+\alpha_2+s\alpha_5) d(x_{2n}, x_{2n+1}). \qquad (20)$$

On the other side and by symmetry we have

$$\begin{aligned}
d(x_{2n+2}, x_{2n+1} &= (g x_{2n+1}, f x_{2n}) \\
&\preceq \alpha_1 d(x_{2n+1}, x_{2n}) + \alpha_2 d(x_{2n+1}, g x_{2n+1}) \\
&\quad +\alpha_3 d(x_{2n}, f x_{2n}) + \alpha_4 d(x_{2n}, g x_{2n+1}) \\
&\quad +\alpha_5 d(x_{2n+1}, f x_{2n}) + \alpha_6 d(g x_{2n+1}, f x_{2n}) \\
&\preceq \alpha_1 d(x_{2n+1}, x_{2n}) + \alpha_2 d(x_{2n+1}, x_{2n+2}) \\
&\quad +\alpha_3 d(x_{2n}, x_{2n+1}) + \alpha_4 d(x_{2n}, x_{2n+2}) \\
&\quad \alpha_5 d(x_{2n+1}, x_{2n+1}) + \alpha_6 d(x_{2n+1}, x_{2n+2}) \\
&\preceq (\alpha_1+\alpha_3) d(x_{2n}, x_{2n+1}) + \alpha_2 d(x_{2n+1}, x_{2n+2}) \\
&\quad +s\alpha_4 (d(x_{2n}, x_{2n+1}) + (x_{2n+1}, x_{2n+2})) + \alpha_6 d(x_{2n+1}, x_{2n+2}) \\
&= (\alpha_1+\alpha_3+s\alpha_4) d(x_{2n}, x_{2n+1}) \\
&\quad +(\alpha_2\alpha_6+s\alpha_4) d(x_{2n+1}, x_{2n+2}). \qquad (21)
\end{aligned}$$

Hence,

$$(1-(\alpha_2+\alpha_6+s\alpha_4)) d(x_{n+2}, x_{2n+1}) \preceq (\alpha_1+\alpha_3+s\alpha_4) d(x_{2n}, x_{2n+1}). \qquad (22)$$

Adding inequalities (20) and (22), we get

$$d(x_{2n+1}, x_{2n+2}) \preceq \frac{2\alpha_1+\alpha_2+\alpha_3+s\alpha_4+s\alpha_5}{2-(\alpha_2+\alpha_3+2\alpha_6+s\alpha_4+s\alpha_5)} d(x_{2n}, x_{2n+1}) = \lambda d(x_{2n}, x_{2n+1}), \qquad (23)$$

where $\lambda = (2\alpha_1 + \alpha_2 + \alpha_3 + s\alpha_4 + s\alpha_5)(2 - (\alpha_2 + \alpha_3 + 2\alpha_6 + s\alpha_4 + s\alpha_5)) < 1/s$. Similarly, it can be shown that

$$d(x_{2n+3}, x_{2n+2}) \preceq d(x_{2n+2}, x_{2n+1}). \tag{24}$$

Therefore,

$$d(x_{n+1}, x_{n+2}) \preceq d(x_n, x_{n+1}). \tag{25}$$

According to Lemma 3.2, we have $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$. Suppose that f is continuous. Then $x^* = \lim x_{n+1} = \lim f^n x_0 = f \lim(f^{n-1}x_0) = f(\lim f^{n-1}x_0) = f(\lim x_n) = f(x^*)$. Therefore, x^* is fixed point of f. Now, we need to show that x^* is a fixed point of g. Since $x^* \subseteq x^*$, we can find the inequality (18) for $x = y = x^*$. Then we have

$$\begin{aligned} d(fx^*, gx^*) &\preceq \alpha_1 d(x^*, x^*) + \alpha_2 d(x^*, fx^*) + \alpha_3 d(x^*, gx^*) \\ &\quad +\alpha_4 d(x^*, fx^*) + \alpha_5 d(x^*, gx^*) + \alpha_6 d(fx^*, gx^*) \\ &= \alpha_1 d(x^*, x^*) + \alpha_2 d(x^*, x^*) + \alpha_3 d(x^*, gx^*) \\ &\quad +\alpha_4 d(x^*, x^*) + \alpha_5 d(x^*, gx^*) + \alpha_6 d(x^*, gx^*) \\ &= \alpha_3 d(x^*, gx^*) + \alpha_5 d(x^*, gx^*) + \alpha_6 d(x^*, gx^*) \\ &= (\alpha_3 + \alpha_5 + \alpha_6) d(x^*, gx^*). \end{aligned} \tag{26}$$

Hence,

$$d(x^*, gx^*) \preceq (\alpha_3 + \alpha_5 + \alpha_6) d(x^*, gx^*). \tag{27}$$

Since $(\alpha_3 + \alpha_5 + \alpha_6) < 1$. Lemma 2.10(1) shows that $(x^*, gx^*) = \theta$; that is, $x^* = gx^*$. Therefore x^* is a fixed point of g. Therefore, f and g have a common fixed point. The proof is similar when g is a continuous mapping.

Theorem 3.6 : Let $(X, \subseteq)$ be a partially ordered set and suppose that there exists a cone b-metric d in X such that the cone b-metric space (X, d) is complete with the coefficient $s \geq 1$ relatvie to a solid cone P. Let $f, g : X \to X$ be two weakly increasing mappings with respect to $\subseteq$. Suppose that the following three conditions holds:

(i) there exist $\alpha_i, i = 1, \cdots, 6$ such that $2s\alpha_1 + 2\alpha_6 + (s+1)(\alpha_2 + \alpha_3) + (s^2 + s)$ $(\alpha_4 + \alpha_5) < 2$ with $\sum\limits_{i=1}^{6} \alpha_i < 1$,

$$\begin{aligned} d(fx, gy) &\preceq \alpha_1 d(x, y) + \alpha_2 d(x, fx) + \alpha_3 d(y, gy) \\ &\quad +\alpha_4 d(y, fx) + \alpha_5 d(x, gy) + \alpha_6 d(fx, gy), \end{aligned} \tag{28}$$

for all comparativbe $x, y \in X$;

(ii) if an increasing sequence $\{x_n\}$ converges to x in X, then $x_n \subseteq x$ for all n. Then f and g have a common fixed point $x^* \in X$.

Proof : As in Theorem 3.3, we can construct an non-decereasing $\{x_n\}$ and prove that there exists $x^* \in X$ such that $x_n \to x^*$, also; by the construction of x_n, $gx_n \to x^*$.

Now condition (iii) implies $x_n \subseteq x^*$ for all n. Putting $x = x^*$ and $y = x_n$ in (28), we get

$$\begin{aligned} d(fx^*, gx_n) &\preceq \alpha_1 d(x^*, x_n) + \alpha_2 d(x^*, fx^*) + \alpha_3 d(x_n, gx_n) \\ &\quad +\alpha_4 d(x_n, fx^*) + \alpha_5 d(x^*, gx_n) + \alpha_6 d(fx^*, gx_n) \\ &= \alpha_1 d(x_n, x^*) + \alpha_2 d(fx^*, x^*) + \alpha_3 d(x_n, gx_n) \\ &\quad +\alpha_4 d(x_n, fx^*) + \alpha_5 d(gx_n, x^*) + \alpha_6 d(fx^*, gx_n) \\ &\preceq \alpha_1 d(x_n, x^*) + \alpha_2 (d(fx^*, gx_n) + d(gx_n, x^*)) \\ &\quad +\alpha_3 (d(x_n, x^*) + x^*, gx_n)) + \alpha_4 (d(x_n, x^*) + (x^*, gx_n) \\ &\quad +d(gx_n, fx^*)) + \alpha_5 d(gx_n, x^*) + \alpha_6 d(fx^*, gx_n) \\ &= (\alpha_1 + \alpha_3 + \alpha_4) d(x_n, x^*) \\ &\quad +(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) d(gx_n, x^*) + (\alpha_2 + \alpha_4 + \alpha_6) d(fx^*, gx_n). \end{aligned} \tag{29}$$

Hence,

$$d(fx^*, gx_n) \preceq \frac{\alpha_1 + \alpha_2 + \alpha_4}{1 - (\alpha_2 + \alpha_4 + \alpha_6)} d(x_n, x^*) + \frac{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}{1 - (\alpha_2 + \alpha_4 + \alpha_6)} d(gx_n, x^*) \tag{30}$$

Since $x_n \to x^*$ and $gx_n \to x^*$, then by Definition 2.9(1) and for $c \gg \theta$ there exists $N_0 \in N$ such that for all $N > N_0, d(x_n, x^*) \ll c(1 - (\alpha_2 + \alpha_4 + \alpha_6))/2(\alpha_1 + \alpha_3 + \alpha_4)$

and $d(gx_n, x^*) \ll c(1-(\alpha_2+\alpha_4+\alpha_6))/2(\alpha_2+\alpha_3+\alpha_4+\alpha_5)$. Then we have

$$\begin{aligned}
d(gx_n, fx^*) &= d(fx^*, gx_n) \\
&\preceq \frac{\alpha_1+\alpha_3+\alpha_4}{1-(\alpha_2+\alpha_4+\alpha_6)} d(x_n, x^*) \\
&\quad + \frac{\alpha_2+\alpha_3+\alpha_4+\alpha_5}{1-(\alpha_2+\alpha_4+\alpha_6)} d(gx_n, x^*) \\
&\ll \frac{\alpha_1+\alpha_3+\alpha_4}{1-(\alpha_2+\alpha_4+\alpha_6)} \times \frac{c}{2} \frac{(1-(\alpha_2+\alpha_4+\alpha_6))}{2(\alpha_1+\alpha_3+\alpha_4)} \\
&\quad + \frac{\alpha_2+\alpha_3+\alpha_4+\alpha_5}{1-(\alpha_2+\alpha_4+\alpha_6)} \times \frac{c}{2} \frac{(1-(\alpha_2+\alpha_4+\alpha_6)}{(\alpha_2+\alpha_3+\alpha_4+\alpha_5)} \\
&= \frac{c}{2} + \frac{c}{2} \\
&= C. \qquad (31)
\end{aligned}$$

Now again, according to Definition 2.9(1) it follows that $gx_n = fx^*$. It follows that $fx^* = x^*$. In a similar way and using that $x^* \subseteq x^*$, we can prove that $gx^* = x^*$. Therefore, f and g have a common fixed point.

Now, we present two examples to illustrate our results.

Example 3.7 : Let $X = [0, 1]$ endowed with the standard order and $E = R^2$ and let $P = \{(x, y) : x, y \geq 0\}$. Define $d : X \times X \to E$ as in Example 2.8. Define $f : X \to X$ by $f(x) = x^2/3$. Then f is continuous and non increasing mappings with respect to $\subseteq$. Then we have

$$\begin{aligned}
d(fx, fy) &= d\left(\frac{x^2}{3}, \frac{y^2}{3}\right) \\
&= \left(\left|\frac{x^2}{3} - \frac{y^2}{3}\right|^2, \left|\frac{x^2}{3} - \frac{y^32}{3}\right|^2\right) \\
&= \frac{1}{9}|x+y|^2(|x-y|^2, |x-y|^2) \qquad (32) \\
&\preceq \frac{4}{9}(|x-y|^2, |x-y|^2) \\
&\preceq \frac{4}{9} d(x, y)
\end{aligned}$$

where $\alpha_1 = 4/9, \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$. It is clear that the conditions of Theorem 3.3 are satisfied. Therefore, f has a fixed point $x = 0$.

Example 3.8 : Let $X = [0, \infty)$ endowed with the standard order and $E = C^1_R[0, 1]$ with $\|u\| = \|u\|_\infty + \|u'\|_\infty, u \in E$ and let $P = \{u \in: u(t) \geq 0$ on $[0, 1]$. It is sell known that this cone is solid, but it is not normal. Define a cone metric $d : X \times X \to E$ by $d(x, y)(t) = |x-y|^2 e^t$. Then (X, d) is a complete cone b-metric space with the coefficient $s = 2$. Let us define $f : X \to X$ by $f(x) = x/4$. Then f is a continuous and increasing mapping with respect to $\subseteq$. Then we have f is an increasing mapping; also we have

$$\begin{aligned} d(fx, fy)(t) &= \left|\frac{1}{4}x - \frac{1}{4}y\right|^2 e^t \\ &= \frac{1}{16}|x-y|^2 e^t \\ &\preceq \frac{1}{16}|x-y|^2 e^t + \frac{1}{17}\left|\frac{x}{4}\right|^2 e^t \\ &\preceq \frac{1}{16}|x-y|^2(t) + \frac{1}{17}d(fx, x)(t) \end{aligned} \tag{33}$$

where $\alpha_1 = 1/16, \alpha_2 = 1/17, \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$.
$\alpha_1 = 16, \alpha_2 = 17, \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$. It is clear that the conditions of Theorem 3.3 are satisfied. Therefore, f has a fixed point $x = 0$.

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